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LETTER TO THE EDITOR

Continuous symmetries of the Lorenz model and the Rikitake two-disc dynamo system

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Abstract. Continuous symmetries of the Lorenz model and the Rikitake two-disc dynamo are given. From these symmetries we derive constants of motion.

Both the Lorenz model (Lorenz 1963)

$$\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= -y - x(z-r) \qquad \sigma, r, b \in \mathbb{R} \\ \dot{z} &= xy - bz \end{aligned} \tag{1}$$

and the Rikitake two-disc dynamo system (Cook and Roberts 1970)

$$\begin{aligned} \dot{x} &= yz - \mu x \\ \dot{y} &= (z-a)x - \mu y \qquad \mu, a \in \mathbb{R} \\ \dot{z} &= 1 - xy \end{aligned} \tag{2}$$

show chaotic behaviour for a wide range of their parameters. However, for various values of the parameters there are (global) symmetry generators. From these symmetry generators we can derive global constants of motion.

In the present letter we give continuous symmetries for the Lorenz model and the Rikitake two-disc dynamo.

The following theorem (Steeb 1982) will be used for deriving constants of motion.

Theorem 1. Let X, V be two vector fields defined on an orientable smooth manifold M. Assume that $[X, V] = fV$ (f: $M \to \mathbb{R}$). Then $L_V(f + \operatorname{div} X) = -f \operatorname{div} V + L_X \operatorname{div} V$.

The vector field V describes the dynamical system. $L_V(\cdot)$ denotes the Lie derivative. X is called a symmetry generator of V. If $-f \operatorname{div} V + L_X \operatorname{div} V = 0$, then $L_V(f + \operatorname{div} X) = 0$ and therefore $f + \operatorname{div} X$ is a constant of motion.

Let us first consider the Lorenz model. Since we are interested in time-dependent constants of motion we extend the autonomous system (1) to the following autonomous system

$$\begin{aligned} \mathrm{d}x/\mathrm{d}\varepsilon &= \sigma(y-x) \qquad \mathrm{d}y/\mathrm{d}\varepsilon = -y - x(z-r) \\ \mathrm{d}z/\mathrm{d}\varepsilon &= xy - bz \qquad \mathrm{d}t/\mathrm{d}\varepsilon = 1. \end{aligned} \tag{3}$$

The associated vector field V defined on $M = \mathbb{R}^4$ is given by

$$V = \sigma(y - x)\partial/\partial x + [-y - x(z - r)]\partial/\partial y + (xy - bz)\partial/\partial z + \partial/\partial t. \tag{4}$$

For the vector field X we make the ansatz

$$X = g(x, y, z)\mathrm{e}^{ct}\partial/\partial t \tag{5}$$

where $c \in \mathbb{R}$ and

$$g(x, y, z) = \sum_{k=0}^{K} \sum_{m=0}^{M} \sum_{n=0}^{N} c_{kmn} x^k y^m z^n \tag{6}$$

with $c_{kmn} \in \mathbb{R}$. The condition that X is a symmetry generator of V, namely $[X, V] = fV$, leads to the following two cases.

Case 1. If $b = 2\sigma$ (r arbitrary), then $[X, V] = 0$, where $X = (x^2 - 2\sigma z)\mathrm{e}^{2\sigma t}\partial/\partial t$. Thus X is a symmetry generator for this case. Theorem 1 leads to the constant of motion

$$f + \operatorname{div} X = \operatorname{div} X = 2\sigma(x^2 - 2\sigma z)\,\mathrm{e}^{2\sigma t} \tag{7}$$

since $f = 0$ and $\operatorname{div} V$ is a constant.

Case 2. If $b = 1$ and $r = 0$ (σ arbitrary), then $[X, V] = 0$, where $X = (y^2 + z^2)\mathrm{e}^{2t}$. Again theorem 1 leads to the constants of motion

$$f + \operatorname{div} X = \operatorname{div} X = 2(y^2 + z^2)\,\mathrm{e}^{2t}. \tag{8}$$

The constants of motion described above have also been given by Segur (1980). In both cases the dynamical system does not show chaotic behaviour. The connection with the Painlevé property has been discussed by Tabor and Weiss (1981).

Consider now the Rikitake two-disc dynamo system. Again we are interested in time-dependent symmetry generators. Thus we consider as described above the extended dynamical system with the associated vector field

$$V = (yz - \mu x)\partial/\partial x + [(z - a)x - \mu y]\partial/\partial y + (1 - xy)\partial/\partial z + \partial/\partial t. \tag{9}$$

We apply the same technique as described above and find: if $a = 0$ (μ arbitrary), then $[X, V] = 0$ where $X = (x^2 - y^2)\mathrm{e}^{2\mu t}\partial/\partial t$. Thus X is a symmetry generator for this case. The theorem given above leads to the constant of motion $(x^2 - y^2)\mathrm{e}^{2\mu t}$. In this case the dynamical system (2) does not behave chaotically.

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